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A HIGHER-ORDER LINEAR THEORY FOR ISOTROPIC PLATES—II. NUMERICAL REALIZATION

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Abstract—The paper presents a numerical technique for a higher-order linear theory for isotropic plates [see Blocki (1992), *Int. J. Solids Structures* **29**(7), 825–836], whereby the natural frequencies of free vibration of a circular "moderately" thick disc of varying thickness profile may be determined when the disc is subjected to the centrifugal loading. The in-plane stress level, arising from rotational effects, is determined by means of a spline interpolation technique. The results of the analysis are compared with the other numerical solutions for thin and moderately thick circular plates.

1. INTRODUCTION

It is well known that the classical thin plate theory neglects the transverse shear strain. So, the solution based on this theory underestimates the deflection and overestimates the natural frequencies. Nevertheless, the obtained results, for the plates of thickness to span ratios less than 0.05 are acceptable for most engineering applications. For thick plates, a theory which considers the shear effect should be used. The aim of this paper is to present numerical results of a new model of the higher-order linear theory for isotropic circular "moderately" thick plates [see Blocki (1992)]. The results of the analysis are compared with the other numerical solutions for thin and moderately thick plates and the exact values obtained by using the frequency equation derived by Mindlin and Deresiewicz (1954).

2. FORMULATION OF THE PROBLEM

Let us suppose that the coordinate system $(0, X_k)$ with the origin 0 at the centre of mass of the disc is motionless and inertial. For the region of disc B the local set of coordinates is associated θ. The dependence between the local and global sets of coordinates is:

$$\mathbf{X} = \mathbf{R}\theta.$$

The local non-inertial set of coordinates for the disc is $(0, r, \psi, x_1)$. This set is rotating with the disc with the angular velocity around the axis of revolution X_1.

Having the sets of local coordinates we can define the regions B:

$$B = \Pi \times (-h, h),$$

where h is the thickness of the disc and Π is the middle plane of the disc.

2.1. *Model of the disc*

The discs are made of the Hook material

$$\mathbf{T} = \mathbf{C} \times \mathbf{E}, \tag{1}$$

where $\mathbf{C}$ is the tensor of elasticity and $\mathbf{E}$ is the Green deformation tensor.

The displacements of the plate at the coordinates (r, ψ, x_1) can be expressed in the form [from Part I of this paper, Blocki (1992)]:

$$u_{d\alpha} = u_{d\alpha}^0 + x_1 \varphi_{d\alpha} + \frac{x_1^3}{6} \chi_{d\alpha}, \quad (\alpha = r, \psi), \quad u_{d1} = u_{d1}^0, \tag{2}$$

where

$$\chi_{dr} = -\frac{8}{h^2}(1-0.5\nu)\left(\frac{\partial u_{d1}^0}{\partial r} + \varphi_{dr}\right),$$

$$\chi_{d\psi} = -\frac{8}{h^2}(1-0.5\nu)\left(\frac{\partial u_{d1}^0}{r\,\partial \psi} + \varphi_{d\psi}\right). \tag{3}$$

In matrix notation:

$$\mathbf{u}_d(\mathbf{r}, \psi, x_1) = \mathbf{U}_d(r, \psi, x_1)\mathbf{q}_d(r, \psi),$$
$$\mathbf{u}_d = \mathrm{col}\,(u_{dr}, u_{d\psi}, u_{d1}),$$
$$\mathbf{q}_d = \mathrm{col}\,(u_{dr}^0, u_{d\psi}^0, u_{d1}^0, \varphi_{dr}, \varphi_{d\psi}), \tag{4}$$

$$\mathbf{U}_d = \begin{bmatrix} 1 & 0 & a\dfrac{\partial}{\partial r} & x_1 + a & 0 \\ 0 & 1 & a\dfrac{\partial}{r\,\partial \psi} & 0 & x_1 + a \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{5}$$

where $a = -(8x_1^3/h^2 6)(1-0.5\nu)$.

2.2. *Equations of motion*

In order to find an approximate solution of the free vibration of the disc the Hamilton principle could be used:

$$\delta \int_{t_1}^{t_2} (\mathscr{E} - \mathscr{K})\,\mathrm{d}t = \int_{t_1}^{t_2} \delta(\mathscr{W})\,\mathrm{d}t, \tag{6}$$

where $\mathscr{E}$, i and $\mathscr{K}$ are elastic and kinetic energies:

$$\mathscr{E} = 0.5 \int_B \mathbf{C} \times (\mathbf{E} \otimes \mathbf{E})\,\mathrm{d}V, \quad \mathscr{K} = 0.5 \int_B \rho_1 \dot{\mathbf{u}}\dot{\mathbf{u}}\,\mathrm{d}V, \tag{7}$$

$\delta\mathscr{W}$ is the variation of external work

$$\delta\mathscr{W} = \int_B \rho_1 \mathbf{b}\,\delta\mathbf{u}\,\mathrm{d}V$$

where **b** is the tensor of body forces.

2.2.1. *The elastic energy*. The Green deformation tensor for the thick disc in two-dimensional theory can be written:

$$\mathbf{E} = \begin{bmatrix} \varepsilon_r & \varepsilon_{r\psi} & \varepsilon_{1r} \\ \varepsilon_{r\psi} & \varepsilon_{\psi} & \varepsilon_{1\psi} \\ \varepsilon_{1\psi} & \varepsilon_{1\psi} & 0 \end{bmatrix},$$

for the Hook material we obtained the elastic energy

$$\mathscr{E} = 0.5 \int_{r_0}^{R} \int_0^{2\pi} \underset{\sim}{\varepsilon}_d^T \mathbf{K} \underset{\sim}{\varepsilon}_d r \, \mathrm{d}r \, \mathrm{d}\phi, \tag{8}$$

where

$$\mathbf{K} = \begin{bmatrix} \frac{E}{1-v^2} & \frac{vE}{1-v^2} & & & \\ \frac{vE}{1-v^2} & \frac{E}{1-v^2} & & & \\ & & 5Gh/6 & & \\ & & & 5Gh/6 & \\ & & & & Gh \end{bmatrix},$$

$$\underset{\sim}{\varepsilon}_d = \mathrm{col}\,(\varepsilon_r, \varepsilon_\psi, \varepsilon_{r1}, \varepsilon_{\psi 1}, \varepsilon_{r\psi}),$$

where

$$\varepsilon_i = \varepsilon_i^0 + x_1 \varkappa_i + \frac{x_1^3}{3!}\mu_i, \quad i = (r, \psi),$$

$$\varepsilon_{r\psi} = \varepsilon_{r\psi}^0 + x_1 \varkappa_{r\psi} + \frac{x_1^3}{3!}\mu_{r\psi},$$

$$\varepsilon_{i1} = \frac{5}{4}\left[1 - \left(\frac{2x_1}{h}\right)^2\right]\varepsilon_{i1}^0,$$

$$\varepsilon_r^0 = \partial u_{dr}/\partial r, \quad \varepsilon_\psi^0 = \partial u_{d\psi}/r\,\partial\psi + u_{d\psi}/r,$$

$$\varepsilon_{r\psi}^0 = \partial u_{d\psi}/\partial r + \partial u_{dr}/r\,\partial\psi - u_{d\psi}/r,$$

$$\varepsilon_{r1}^0 = \partial u_{d1}/\partial r + \varphi_{dr}, \quad \varepsilon_{\psi 1}^0 = \partial u_{d1}/r\,\partial\psi + \varphi_{d\psi},$$

$$\varkappa_r = \partial\varphi_{dr}/\partial r, \quad \varkappa_\psi = \partial\varphi_{d\psi}/r\,\partial\psi + \varphi_{dr}/r,$$

$$\varkappa_{r\psi} = \partial\varphi_{d\psi}/\partial r + \partial\varphi_{dr}/r\,\partial\psi - \varphi_{d\psi}/r,$$

$$\mu_r = \partial\chi_{dr}/\partial r, \quad \mu_\psi = \partial\chi_{d\psi}/r\,\partial\psi + \chi_{dr}/r,$$

$$\mu_{r\psi} = \partial\chi_{d\psi}/\partial r + \partial\chi_{dr}/r\,\partial\psi - \chi_{d\psi}/r.$$

2.2.2. *The kinetic energy*. The kinetic energy can be written as:

$$\mathscr{K} = 0.5 \int_{r_0}^{R} \int_0^{2\pi} \mathbf{q}_{d1}^{\mathrm{T}} \mathbf{M} \mathbf{q}_{d1} r \, \mathrm{d}r \, \mathrm{d}\phi, \tag{9}$$

where

$$\mathbf{q}_{d1} = \mathrm{col}\,(u^0_{dr}, u^0_{d\psi}, u^0_{d1}, \varphi_{dr}, \varphi_{d\psi}, \chi_{dr}, \chi_{d\psi}),$$

$$\mathbf{M} = \rho\begin{bmatrix} h & & & & & & \\ & h & & & & & \\ & & h & & & & \\ & & & \frac{h^3}{12} & & -\frac{h^5}{480} & \\ & & & & \frac{h^3}{12} & & -\frac{h^5}{480} \\ & & & -\frac{h^5}{480} & & \frac{h^7}{16128} & \\ & & & & \frac{h^5}{480} & & \frac{h^7}{16128} \end{bmatrix}.$$

2.2.3. *The external work.* The variations of external work $\delta\mathscr{W}$ caused by the rotatory motion can be written in the form

$$\delta W_1 = \int_{r_0}^{R}\int_0^{2\pi} \mathbf{A}_\Omega \mathbf{q}_{d1}\,\delta\mathbf{q}_{d1}, \tag{10}$$

where

$$\mathbf{A}_\Omega = 0.5\rho\Omega^2\begin{bmatrix} h & & & & & & \\ & h & & & & & \\ & & 0 & & & & \\ & & & \frac{h^3}{12} & & -\frac{h^5}{480} & \\ & & & & \frac{h^3}{12} & & -\frac{h^5}{480} \\ & & & -\frac{h^5}{480} & & \frac{h^7}{16128} & \\ & & & & \frac{h^5}{480} & & \frac{h^7}{16128} \end{bmatrix}.$$

The dependence between $\mathbf{q}_{d1}$ and $\mathbf{q}_d$ can be found from eqn (3):

$$\mathbf{q}_d = \mathbf{Q}\mathbf{q}_{d1},$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ & & a_1\frac{\partial}{\partial r} & a_1 & \\ & & a_1\frac{\partial}{r\,\partial\psi} & & a_1 \end{bmatrix},$$

$$a_1 = -\frac{8}{h^2 6}(1-0.5\nu).$$

$$\mathbf{q}_{\mathrm{d}} = \mathrm{col}\,(u^0_{\mathrm{d}r}, u^0_{\mathrm{d}\psi}, u^0_{\mathrm{d}1}, \varphi_{\mathrm{d}r}, \varphi_{\mathrm{d}\psi}),$$

$$\mathbf{q}_{\mathrm{d}1} = \mathrm{col}\,(u^0_{\mathrm{d}r}, u^0_{\mathrm{d}\psi}, u^0_{\mathrm{d}1}, \varphi_{\mathrm{d}r}, \varphi_{\mathrm{d}\psi}, \chi_{\mathrm{d}r}, \chi_{\mathrm{d}\psi}).$$

2.2.4. *The stresses in the middle plane of the disc.* The additional strain energy in bending due to initial in-plane stress $\tau_{r0}, \tau_{\psi 0}$ caused by the static components of centrifugal force is

$$\mathbf{A}_{\mathrm{dc}} = 0.5\int_{r_0}^{R}\int_{0}^{2\pi}\left[N_r\left(\frac{\partial u_{\mathrm{d}1}}{\partial r}\right)^2 + N_\theta\left(\frac{\partial u_{\mathrm{d}1}}{r\,\partial\psi}\right)^2\right] r\,\mathrm{d}r\,\mathrm{d}\phi, \tag{11}$$

where

$$\mathbf{N}_r = \int_{-h/2}^{h/2}\tau_{r0}\,\mathrm{d}x_1, \quad \mathbf{N}_\psi = \int_{-h/2}^{h/2}\tau_{\psi 0}\,\mathrm{d}x_1.$$

The stresses $\tau_{r0}, \tau_{\psi 0}$ in radial and angular directions are calculated by means of a spline interpolation technique (Irie *et al.*, 1979; Rzadkowski, 1990).

2.3. *The functional of the problem*

The functional of the problem can be written as:

$$\delta\mathscr{P} = \delta\int(\mathscr{E}_{\mathrm{d}} - \mathbf{T}_{\mathrm{d}} - \mathbf{A}_{\mathrm{d}\Omega} + \mathbf{A}_{\mathrm{dc}})\,\mathrm{d}t = 0. \tag{12}$$

The parameters describing vibration of the disc $u^0_{\mathrm{d}r}, u^0_{\mathrm{d}\psi}, u^0_{\mathrm{d}1}, \varphi_{\mathrm{d}r}, \varphi_{\mathrm{d}\psi}$ were approximated by

$$I_{\mathrm{d}} = \sum_{j=1}^{C}\sum_{k=1}^{K}\varphi_{I\mathrm{d}j}(r)(a_{jk}\sin k\psi + b_{jk}\cos k\psi)\sin pt = \sum_{k=1}^{K}\mathbf{A}_{I\mathrm{d}}(\mathbf{a}\sin k\psi + \mathbf{b}\cos k\psi)\sin pt, \tag{13}$$

where $\varphi_{I\mathrm{d}j}(r)$ are the eigenfunctions of the cantilever beam. For example:

$$u_{\mathrm{d}1} = \sum_{j=1}^{C}\sum_{k=1}^{K}\varphi_{u\mathrm{d}1j}(r)(a_{jk}\sin k\psi + b_{jk}\cos k\psi)\sin pt,$$

$$\varphi_{u\mathrm{d}1j} = [\cosh c_j(r-r_0)/(R-r_0) - \cos c_j(r-r_0)/(R-r_0)]$$
$$-\alpha_j[\sinh c_j(r-r_0) - \sin c_j(r-r_0)/(R-r_0)], \quad \cosh c_k \cos c_k + 1 = 0,$$

$$\alpha_j = (\cos c_j + \cosh c_j)/(\sin c_j + \sinh c_j),$$

$$\varphi_{u\mathrm{d}rj} = \sin \varkappa_j(r-r_0)/(R-r_0),$$

$$\varphi_{u\mathrm{d}\psi j} = \{(r-r_0)/(R-r_0),\ j=0, \quad \cos\varkappa_j(r-r_0)/(R-r_0),\ j=1,\ldots,k\},$$

$$\varkappa_j = (2j-1)\pi/2.$$

Substituting eqns (8)–(11) and (13) into (12) we obtain

$$(\mathbf{K} - p^2\mathbf{M})y = 0.$$

3. NUMERICAL RESULTS

The presented theory is a higher-order plate theory by comparison with the Mindlin theory. In order to compare the numerical results obtained by these two theories the natural frequencies for the simplified theory, which is equivalent to the Mindlin plate theory, were

Table 1. Eigenvalues of a free-clamped annular uniform "moderately" thick plate $\nu = 0.3$, $\beta = r_0/R = 0.2$, $h_0/R = 0.1$, $\rho = 0.78\ 10^4$ kg m^{-3}, $E = 0.207\ 10^{12}$ N m^{-2}, $m = 0$

Błocki theory	2.258	5.465	8.97	11.84
Simplified theory	2.254	5.427	8.87	11.777
Mindlin and Deresiewicz (1954)	2.254	5.413	8.828	11.782
Classical	2.276	5.683	9.70	13.656

Table 2. Comparison of non-dimensional natural frequencies, for a static "moderately" thick plate, $h_0/R = 0.1$, $r_0/R = 0.2$, $\nu = 0.3$, $h_1/h_0 = 0.5$. Upper values are those obtained from Ritz analysis, lower values those obtained from Irie *et al.* (1980b). Mode $m = 0$ $\rho = 0.78\ 10^4$ kg m^{-3}, $E = 0.027\ 10^{12}$ N m^{-2}, $\nu = 0.3$

Linear	2.281	5.034	8.022	Błocki
	2.279	5.011	7.963	Irie *et al.* (1980).
Exponential	2.243	4.938	7.885	
	2.241	4.916	7.828	

Linear $h = h_0\,(1-(1-h_1/h_0))(r-r_0)/(R-r_0)$.
Exponential $h = h_0(h_1/h_0)^{(r-r_0)/(R-r_0)}$.

also calculated. For simplified theory the displacements of the plate at the coordinates (r, ψ, x_1) can be expressed in the form:

$$u_{d\alpha} = u_{d\alpha}^0 + x_1 \varphi_{d\alpha} \quad (\alpha = r, \psi),$$
$$u_{d1} = u_{d1}^0. \tag{14}$$

The eigenvalues of free vibration of a free-clamped uniform "moderately" thick plate were shown in Table 1. Upper values are those obtained by the author, middle values those obtained from the simplified theory (eqn 14), lower exact values were obtained by using the frequency equation derived by Mindlin and Deresiewicz (1954).

In Table 1, the eigenvalues of the "moderately" thick plate were also compared with the value obtained by the classical theory, in which neither the rotatory inertia nor the shear deformation were taken into account (Irie *et al.*, 1980a). In general, the eigenvalues of Mindlin plate are smaller than those obtained by the classical theory. The results of Mindlin theory are smaller than those of Błocki theory. The last conclusion confirms the results presented by Niordson (1979). The difference between natural frequencies obtained by Mindlin and Błocki theories decreases when the ratio h_0/R decreases.

Table 2 shows the eigenvalues of a stationary "moderately" thick plate ($h_0/R = 0.1$) of varying thickness. Upper values are those obtained from this analysis, lower values those obtained from Irie *et al.* (1980b) for the Mindlin plate.

Tables 3 and 4 present the numerical results for thin plates.

Table 3 presents the non-dimensional natural frequencies for a thin plate ($h_0/R = 0.01$), of variable profile, $h = h_0(1-\beta(r/b))$ for various values of $\beta = r_0/R$. The upper values are those obtained by the author, middle values those obtained from the finite element method by Kennedy and Gorman (1977) and lower values those obtained from Soni and Amba-Rao (1975) who applied a Chebyshev collocation method.

Table 4 shows the eigenvalues of a rotating thin ($h_0/R = 0.02$) uniform disc. The values in the first column are those obtained from this analysis, in the second those obtained from Irie *et al.* (1979).

4. CONCLUSIONS

Numerical techniques have been developed which enable the natural frequencies of a clamped circular disc to be evaluated for any thickness variation when the disc is subjected to rotation. Convergence of the technique is examined for a varying number of numerical

Table 3. Comparison of non-dimensional natural frequencies, for a static thin disc ($h_0/R = 0.01$) of variable profile, $h = h_0\,(1-\beta(r/b))$ for various values of β upper values are those obtained from Ritz analysis, middle values those obtained from the finite element method by Kennedy and Gorman (1977) and lower values those obtained from Soni and Amba-Rao (1975). Mode $m = 0$, $n = 0$ and $m = 0$, $n = 1$, $\rho = 0.78\ 10^4$ kg m^{-3}, $E = 0.207\ 10^{12}$ N m^{-2}, $\nu = 0.3$, $R = 0.202$ m, $\Omega = 0$, $\lambda_0 = \omega_n\ 12(1-\nu^2)\rho b^4/Eh_0^2$, $r_0/R = 0.1$

β	$m = 0$, $n = 0$	$m = 0$, $n = 1$	
0.7	4.0271	17.7651	Blocki
	4.0317	17.7218	Kennedy and Gorman (1977)
	3.9848	17.4941	Soni and Amba-Rao (1975)
0.5	3.978	20.0111	
	3.9871	20.0185	
	9.9565	19.8647	
0.3	4.0368	22.1474	
	4.0501	22.1851	
	4.0209	22.0155	
0.1	4.1502	24.1950	
	4.1670	24.2585	
	4.1321	24.0323	

Table 4. Comparison of non-dimensional natural frequencies, of a free-clamped rotating uniform thin disc, values in the first column are those obtained from this analysis, in the second those obtained from Irie *et al.* (1979), $\omega = 1901.64$, ($\Omega = 0.1$). The geometrical parameters of the disc are: $r_0 = 0.0254$ m, $R_d = 0.127$ m, $h_0 = 0.00254$ m, $h = h_0$, $\rho = 0.78\ 10^4$ kg m^{-3}, $E = 0.207\ 10^{12}$ N m^{-2}, $\nu = 0.3$, $\Omega = 0.1 = 2(1-\nu^2)/E(h_0/a)\zeta(\gamma/g)\omega^2R_d$, $h_0/R_d = 0.02$, $r_0/R_d = \beta = 0.2$, $\lambda^4 = \gamma h_0 R_d^4 p^2/gD_0$, $D_0 = Eh_0/12(1-\nu^2)$

	Author	Irie *et al.* (1979)
λ_{00}	3.0049	2.994
λ_{01}	6.2023	6.180

results presented in the literature for thin and moderately thick plates, obtaining satisfactory results.

The eigenvalues calculated by the Mindlin plate theory are smaller than those of Blocki theory. That conclusion confirms the results presented by Niordson (1979).

In order to compare these theories the experimental test should be performed on a thick circular plate or the eigenvalues of the free vibration of a thick disc should be calculated using the three-dimensional model of the plate.

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